MULTI-PARTICLE DIFFERENTIAL METHODS IN THE STATICS OF COMPOSITES

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In addition to effective field [1-3] and effective medium [4, 5] methods, the differential effective medium method (EMM) [6-13] is used in the mechanics of composite materials. A comparative analysis of the former two methods was carried out in [2]. Differential EMM arose as an alternative to the ambiguous assumptions of EMM, where the value of the effective modulus is used as an estimate of the average strain concentration in an isolated inclusion. Differential EMM is considered as a process of consecutive additions of the inclusion phase in a uniform medium with a modulus equal to the effective modulus of the medium with the previous additions of inclusions to the matrix. This process can take place in two ways. In the first, it is assumed that the inclusion phase consists of an infinite number of fractions with infinite size differences, and successive accomodations of the inclusions in the corresponding uniform medium leads to ascending order of inclusion size. In the second, inclusions of the initial size are added to the medium in infinitesimal amounts to reach the final real concentration. Both methods give equivalent results, since at each interation step, the same solution to the single-particle problem is used to estimate the effective modulus of the medium with an infinitesimal inclusion concentration. Thus both EMM [6-12] and the effective field method (EFM) [13] are applicable.

In this work, we do not solve the single-particle problem [6-13], but instead solve the multi-particle problem at each iteration step of the differential scheme. To do this we use a previously proposed algorithm [1, 3]. In comparisons of calculations with experimental data, we show the advantage of the multi-particle differential method over the single-particle method.

1. General Relations. Let the uniform matrix $v_0$ with tensor properties $L(x) = L_0 (x \in v_0)$ contain a random set $X = (V_k, x_k, \omega_k)$ $(k = 1, 2, \ldots)$ of ellipsoids $v_k$ with characteristic functions $V_k$, centers $x_k$, which form a Poisson set, with semi-axes $a_k^i$ $(a_k^1 \geq a_k^2 \geq a_k^3)$, the set of Euler angles $\omega_k$ and tensor properties $L(x) \equiv L_0 + L_1(x) = L_0 + L_1^{(k)}(x)$ $(x \in v_k)$, where the tensor $L_1^{(k)}$ is an inhomogeneous function of the coordinates. The concrete sense of the tensor $L(x)$ can vary; in transport problems (electrical conductivity, thermal conductivity), it is the second-rank tensor of transport coefficients. In elastic problems, by $L(x)$ we understand the fourth-rank tensor of elastic moduli. The local equation of state which couples the flux density tensor $\sigma$ and the strain field $\varepsilon$ with potential $u$ is taken in the form

$$\sigma(x) = L(x)\varepsilon(x),\ \ \varepsilon(x) = \nabla u(x) \qquad (1.1)$$

($\nabla$ is the gradient operator and the symmetrized gradient for scalar and vector potentials $u$, respectively). We are using the standard tensor analysis notation. Substituting (1.1) into the equation of equilibrium $\operatorname{div}\sigma(x) = 0$, we obtain a differential equation in terms of the potential $u$. Transforming the latter into integral form, we have [3]

$$\varepsilon(x) = \langle\varepsilon\rangle + \int U(x-y)\left[V(y)L_1(y)\varepsilon(y) - \langle L_1\varepsilon\rangle\right]dy,\ V(y) = \bigcup_{i=1} V_i(y). \qquad (1.2)$$

Here $U = \nabla\nabla G$; $G$ is Green's tensor for the Lamé equation of a uniform medium with tensor properties $L_0$; $\langle(\cdot)\rangle$ is the operation of ensemble averaging of the statistically uniform ergodic field $X$. If the set $X = (V_k, x_k, \omega_k)$ is finite $(k = 1, \ldots, n)$, then for a given

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strain field $\varepsilon_0$ at infinity, (1.2) transform to:

$$\varepsilon(x) = \varepsilon_0 + \int U(x-y)\, V(y)\, L_1(y)\, \varepsilon(y)\, dy, \; V(y) = \sum_{i=1}^{n} V_i(y). \tag{1.3}$$

An estimate of the tensor of effective properties $L^*$ in the expression

$$\langle\sigma\rangle = L^* \langle\varepsilon\rangle \tag{1.4}$$

is determined from the averaged local equation of state (1.1)

$$L^* = L_0 + \sum_{i=1}^{N} \langle\langle R_i^* \rangle\rangle_\omega c_i, \; \langle L_1(y)\, \varepsilon(y)\, V_i(y)\rangle_i \equiv R_i^* \langle\varepsilon\rangle, \tag{1.5}$$

where $\langle\langle(\cdot)_i\rangle\rangle_\omega$ is the operation of averaging over all possible orientations $\omega_i$ of inclusion $v_i$; N is the number of components. The first equation in (1.5) can be rewritten in the form

$$L^* = L_0 + \sum_{i=1}^{N} \langle\langle R_i^* A_0^{-1} \rangle\rangle_\omega \left[ c_0 I + \sum_{j=1}^{N} c_j \langle\langle A_j^0 \rangle\rangle_\omega \right]^{-1}; \tag{1.6}$$

$$\langle\varepsilon\rangle_i = A_i^0 \langle\varepsilon\rangle_0, \; \langle\varepsilon\rangle_0 = A_0 \langle\varepsilon\rangle, \; c_\alpha = \langle V_\alpha \rangle \tag{1.7}$$

(here $\langle(\cdot)\rangle_\alpha$ is the volume average of component $\alpha$ ($\alpha$ = 0, 1, ..., N)).

To compute $R_i^*$ (1.5) we introduce $\phi(v_m|x_1, \ldots, x_n)$, which is the conditional probability density for the location of the m-th inclusion in the region $v_m$ for fixed inclusions $v_1, \ldots, v_n$ with centers $x_1, \ldots, x_n$. We know that for $x_m \in v^0_{1,\ldots,m} = \bigcup v_j^0$ (the union over j = 1, ..., m), the function $\phi(v_m|x_1, \ldots, x_n) = 0$, and that $\phi(v_m|x_1, \ldots, x_n) \to \phi(v_m)$ for $|x_i - x_m| \to \infty$ (i = 1, ..., n). Here the regions $v_j^0 \supset v_j$ (j = 1, ..., n) have characteristic functions $V_j^0$. We average (1.2) over the set X for fixed inclusions $v_1$; $v_1$, $v_2$; ... with the help of the various conditional density distributions $\phi(v_m|x_1, \ldots, x_n)$ (n = 1, 2, ...). We obtain an infinite system of coupled integral equations (j = 1, ..., n):

$$\begin{gathered}\langle\varepsilon(x)\,|\,x_1\rangle - \int U(x-y)\, V_1(y) \langle L_1(y)\, \varepsilon(y)\,|\,x_1\rangle\, dy = \\ = \langle\varepsilon\rangle + \int U(x-y)\, [\langle L_1(y)\, \varepsilon(y)\,|\,y;\, x_1\rangle - \langle L_1 \varepsilon\rangle]\, dy, \\ \langle\varepsilon(x)\,|\,x_1, \ldots, x_j\rangle - \sum_{i=1}^{j} \int U(x-y)\, V_i(y) \langle L_1(y)\, \varepsilon(y)\,|\,x_1, \ldots, x_j\rangle\, dy = \\ = \langle\varepsilon\rangle + \int U(x-y)\, [\langle L_1(y)\, \varepsilon(y)\,|\,y;\, x_1, \ldots, x_j\rangle - \langle L_1 \varepsilon\rangle]\, dy\end{gathered} \tag{1.8}$$

($\langle\cdot|y; x_1, \ldots, x_j\rangle$ is the conditional average for fixed inclusions with centers at y, $x_1$, ..., $x_j$ and $y \neq x_1, \ldots, x_j$). We denote the right-hand side of the j-th (j = 1, 2, ...) line of the system by the field $\hat{\varepsilon}(x)_{1,\ldots,j}$, which has the simple physical meaning of a strain field, in which are located n fixed inclusions. In this case each inclusion $v_i$ from the selected fixed inclusions is located in the field

$$\bar{\varepsilon}_i(x) = \hat{\varepsilon}(x)_{1,\ldots,j} + \sum_{k \neq i} \int U(x-y)\, V_k(y)\, L_1(y)\, \varepsilon(y)\, dy, \; x \in v_i. \tag{1.9}$$

The strain field inside the inclusion $v_i$ depends only on the value of the, generally speaking, nonuniform field $\bar{\varepsilon}_i$ in the region of $v_i$. In order subsequently to omit the dependence of terms in (1.8) on $x \in v_i$, we average each j-th line (j = 1, 2, ...) of (1.8) over the volume $v_i$ of the i-th inclusion (i = 1, ..., j), so that

$$\langle \varepsilon(x) | x_1, \ldots, x_j \rangle_i - \sum_{k=1} \bar{v}_i^{-1} \iint U(x-y) V_i(x) V_k(y) \langle L_1(y) \varepsilon(y) | x_1, \ldots \tag{1.10}$$

$$\ldots, x_j \rangle \, dy \, dx = \langle \varepsilon \rangle + \bar{v}_i^{-1} \iint U(x-y) V_i(x) [\langle L_1(y) \varepsilon(y) | y; x_1, \ldots, x_j \rangle - \langle L_1 \varepsilon \rangle] \, dy \, dx$$

($\langle (\cdot) \rangle_i \equiv \bar{v}_i^{-1} \int (\cdot) V_i(x) \, dx$ is the averaging operation over the volume of the i-th inclusion). Perhaps some assumptions can be made concerning the right-hand side of (1.10), in order to close system (1.10). But in any case, the following steps appear in the solution of the problem for a finite number of inclusions (left-hand part of (1.10)) in the strain field, prescribed or determined from self-consistent estimates.

2. Finite Number of Inclusions in a Uniform Matrix. We solve this auxiliary problem for a finite set $X_k(V_k, x_k, \omega_k)$ $(k = 1, \ldots, n)$ of ellipsoids in a uniform matrix with tensor properties $L_0$ and a given field $\langle \hat{\varepsilon}(x)_{1,\ldots,n} \rangle$ at infinity. For an approximate solution to the problem, we use one of the MEF hypotheses [1-3], according to which each inclusion $v_i$ $(i = 1, \ldots, n)$ is located in a uniform field $\bar{\varepsilon}_i$. The error incurred with such an assumption was estimated in [3] for an elastic problem of a plane with right-angle cuts. Then the right-handed side of (1.10) becomes algebraic for fixed inclusions $\langle \widehat{\varepsilon}(x)_{1,\ldots,n} \rangle_i$ $(x \in v_i)$:

$$\langle \bar{\varepsilon}(x) | x_1, \ldots, x_n \rangle_i - \sum_{j=1}^{n} \bar{v}_i^{-1} \iint U(x-y) V_j(y) V_i(x) \times \langle L_1(y) \varepsilon(y) | y; x_1, \ldots, x_n \rangle_j \, dx \, dy = \langle \widehat{\varepsilon}(x)_{1,\ldots,n} \rangle; \tag{2.1}$$

$$\langle \varepsilon(x) \rangle_i - \langle U(x) \rangle_i \langle L_1(x) \varepsilon(x) \rangle_i = \langle \bar{\varepsilon}(x) | x_1, \ldots, x_n \rangle_i. \tag{2.2}$$

In deriving (2.2), we used the characteristic property of an ellipsoid $\langle U(x) \rangle_i = \text{const}$ for $x \in v_i$ [4]. Due to the linearity of problem (2.2), there exists a constant tensor $B_i$ such that

$$\langle \varepsilon(x) \rangle_i = B_i \langle \bar{\varepsilon}(x) | x_1, \ldots, x_n \rangle_i; \tag{2.3}$$

$$\langle L_1(x) \varepsilon(x) \rangle_i = R_i \langle \bar{\varepsilon}(x) | x_1, \ldots, x_n \rangle_i \tag{2.4}$$

($R_i = \langle U(x) \rangle_i^{-1} (B_i - I)$). For example, for a uniform ellipsoidal inclusion $v_i$ for $L_1^{(i)} = \text{const}$

$$B_i = (I + P_i L_1^{(i)})^{-1}, \tag{2.5}$$

($P_i = -\int U(x-y) V_i(y) dy$ $(x \in v_i)$ is a constant tensor which does not depend on the physical properties and dimensions (nor on the form) of the ellipsoid $v_i$). The tensor $P_i$ can be represented in the form $P_i = S_i L_0^{-1}$, where the tensor $S_i$ is constructed for a sphere, an elliptical cylinder, and for oblate and prolate ellipses in problems of elasticity (Eshelby tensor) [4] and of conductivity [14]. The tensor $B_i$ (2.3) is also known for a two-layered spheroid [15, 16].

From (1.3) and (2.3) we obtain

$$\langle \bar{\varepsilon}(x) | x_1, \ldots, x_n \rangle_i - \sum_{j \neq i}^{n} \bar{v}_i^{-1} \iint U(x-y) V_i(x) V_j(y) R_j \langle \bar{\varepsilon}(y) | y; x_1, \ldots, x_n \rangle_j \, dx \, dy = \langle \widehat{\varepsilon}(x)_{1,\ldots,n} \rangle_i \, , \tag{2.6}$$

which is a system of linear algebraic equations in $\bar{\varepsilon}_i$. It can be solved using the standard methods of linear algebra. To do this, we switch from the tensor form of writing (2.6) to the matrix form [4]. We form the matrix $Z^{-1}$ with elements $Z^{-1}_{mk}$ ($m, k = 1, \ldots, N$), in the form of a submatrix

$$Z^{-1}_{mk} = I\delta_{mk} - (1 - \delta_{mk}) R_m S(x_m - x_k), \quad (2.7)$$

$$S(x_m - x_k) = \bar{v}^{-1}_m \iint U(x - y) V_m(x) V_k(y)\, dx\, dy.$$

Then the solution to (2.6) is written in the form

$$R_i \langle \bar{\varepsilon}(x) \,|\, x_1, \ldots, x_n \rangle_i = \sum_{j=1}^{n} Z_{ij} R_j \langle \hat{\varepsilon}(x)_{1,\ldots,n} \rangle_j. \quad (2.8)$$

The solution to (2.8) can also be constructed using the method of successive approximations [1]; so, taking into account the first two iterations

$$Z_{ij} = I\delta_{ij} + (1 - \delta_{ij}) \sum_{j \neq i}^{n} S(x_i - x_j) R_j. \quad (2.9)$$

3. The Multi-Particle Effective Medium Method. In the single-particle effective medium method, it is assumed that the isolated inclusion is located in a uniform matrix with tensor properties $L = L^*(c)$ and a given strain field $\langle \varepsilon \rangle$ at infinity. Then for $L^*$ we obtain the implicit equation

$$L^* = L_0 + \sum_{i=1}^{N} \langle\!\langle \bar{R}_i \rangle\!\rangle_\omega c_i, \quad (3.1)$$

$$\bar{R}_i = \langle \bar{U}(x) \rangle_i^{-1} (\bar{B}_i - I), \quad \bar{B}_i = (I + \bar{P}_i (L(x) - L^*))^{-1}, \quad x \in v_i,$$

where the bar over the tensor indicates that it is computed using the fact that the matrix properties coincides with those of the effective medium.

In the proposed variant of multi-particle ($n$-particle) effective field method, we assume that the inclusions are located in a matrix with modulus $L^* = L^*(c)$ and that a uniform field $\langle \varepsilon \rangle$ acts on each of the $n$ inclusions. The last assumption makes it possible to close system (1.10), which, using the solution for one (2.3) and $n$ inclusions is given in the form ($j = 1, \ldots, n - 1$, $i = 1, \ldots, j$)

$$\langle \hat{\varepsilon}(x)_{1,\ldots,j} \rangle_i = \langle \varepsilon \rangle + \int \Big\{ \bar{S}(x_i - x_q) \varphi(v_q | x_1, \ldots, x_j) \sum_{l=1}^{j+1} \bar{Z}_{ql} \bar{R}_l \langle \hat{\varepsilon}(x)_{1,\ldots,j+1} \rangle_l - \bar{S}_i(x_i - x_q) \langle \bar{R} V \rangle \langle \varepsilon \rangle \Big\} dx_q,$$

$$\langle \hat{\varepsilon}(x)_{1,\ldots,n-1} \rangle_i = \langle \varepsilon \rangle + \int \Big\{ \bar{S}(x_i - x_q) \varphi(v_q | x_1, \ldots, x_{n-1}) \sum_{l=1}^{n} \bar{Z}_{ql} \bar{R}_l - \bar{S}_i(x_i - x_q) \langle \bar{R} V \rangle \Big\} dx_q \langle \varepsilon \rangle.$$

From the last equation, we determine the effective field $\langle \hat{\varepsilon}(x)_{1,\ldots,n-1} \rangle_i$ with an accuracy to first order terms in $c$. We substitute this value into the preceding equation and so forth. Thus we obtain a representation for $\bar{R}^*_i$ in the form of a tensor polynomial in $c$ of degree $n$.

4. Scheme of the Differential Method. We examine a generalization of a differential method scheme [12, 13] in the case of a multi-component filler with complex structure. We take the volume $v$ of the composite medium with some finite concentration of inclusions $i = 1, 2, \ldots$ (which in general is different from the concentration $c_i$). The composite medium is replaced by a uniform volume $v$ with tensor properties $L$, determined from the equation

$\langle\sigma\rangle = L\langle\varepsilon\rangle$. The infinitesimal discrete volume of the uniform medium is removed and replaced by the sum of the components i = 1, 2, ... . That is, the representative volume $dv_i$ with properties L is replaced by the same volume $dv_i$ with properties of the i-th component. If $\tilde{R}_i^*$ is the average value of the coefficient of concentration of the polarization tensor

$$\langle\tilde{L}_1(x)\varepsilon(x)\rangle_i \equiv \tilde{R}_i^* \langle\varepsilon\rangle \quad (\tilde{L}_1(x) \equiv L(x) - L)$$

of element $dv_i$, then the increment to the tensor of effective porperties L can be found from (1.5) in the form

$$dL = \sum_{i=1}^{N} \ll \tilde{R}_i^* \frac{dv_i}{v} \gg_\omega. \tag{4.1}$$

Since it is convenient to carry out the calculation with the additional volume portions $dc_i$, it can be shown that [12, 13]

$$\frac{dv_i}{v} = \lambda_i \frac{dc}{1-c}, \quad c = \sum_{i=1}^{N} c_i, \quad \lambda_i = c_i/c$$

and consequently,

$$dL = \sum_{i=1}^{N} \ll \tilde{R}_i^* \lambda_i \frac{dc}{1-c} \gg_\omega. \tag{4.2}$$

Equation (1.7) is an ordinary differential equation with respect to the unknown tensor L with initial conditions $L(0) = L_0$, $c_i(0) = 0$, $c(0) = 0$ (i = 1, ..., N) and independent variable c ($\lambda_i = c_i/c$ = const). The basis for (1.7) for uniform inclusions is given in [8], and for a single-component filler in [5, 6, 10, 11]. To close (1.7), it is necessary to establish a concrete value for $\tilde{R}_i^*$, which can be determined on the basis of single-particle and multi-particle approaches.

5. Single-Particle Differential Method. In the widely used single-particle differential method, it is assumed that the tensor $\tilde{R}_i^*$ is determined from the solution to the problem for an isolated inclusion i in a uniform medium with tensor properties L and for a uniform field $\langle\varepsilon\rangle$ given at infinity. For the postulated equality of $R_i^*$ (1.5) and $\bar{R}_i$ (3.1) ($R_i^* = \bar{R}_i$), the effective field method is valid. Then the use of $\bar{R}_i$ (3.1) in the differential equation for the effective modulus (4.2)

$$\tilde{R}_i^* = \bar{R}_i \tag{5.1}$$

is, in its own way, a combination of the single-particle effective medium method and differential scheme (4.2) [12, 13].

Another version of the single-particle differential method is based on the application of the hypothesis of averaged strains of Mori and Tanaka [5, 17], which is a special case of the hypothesis of the method of effective fields [2]. It follows from just this definition of the tensors $A_i^0$ (i = 0, 1, ...) (1.7), that the tensor $\tilde{R}_i^*$ (4.1) can be put in the form

$$\tilde{R}_i^* = \tilde{R}_i^* A_0^{-1} \left(c_0 I + \sum_{j=1}^{N} \ll c_j A_j^0 \gg_\omega\right)^{-1}. \tag{5.2}$$

In the Mori–Tanaka hypothesis [5, 17, 18], it is assumed that

$$A_i^0 = B_i, \tag{5.3}$$

and to obtain an alternate version of the single-particle differential method, it is sufficient to take

$$\tilde{R}_i^* A_0^{-1} = R_i, \tag{5.4}$$

then (4.2), (5.2)-(5.4) form a closed system for calculation of the effective modulus

$$dL = \sum_i \ll R_i \gg_\omega \Big(c_0 I + \sum_{j=1} \ll c_j B_j \gg_\omega^{-1}\Big)^{-1} \lambda_i \frac{dc}{1-c}. \quad (5.5)$$

Hypothesis (5.3) states that the tensor of the mean strain concentration field $\langle\varepsilon\rangle_i$ in the inclusions with respect to the mean strain field in the matrix $\langle\varepsilon\rangle_0$, is independent of the filler concentration in a real composite. Relation (5.4) additionally confirms the equality of this concentration tensor to the analogous concentration tensor at each step of the iteration process involving the addition of an infinitesimal concentration of inclusions to the uniform medium with tensor properties L(c). Formula (5.5) is a combination of single-particle EFM [2] and the differential scheme (4.2).

For composite media with a single-component uniform filler, the following relation holds:

$$A_i^0 = \frac{1-c_1}{c_1}(L - L_0)(L^{(1)} - L)^{-1}.$$

By substituting this into (5.5) and using the assumptions (5.3), (5.4), we obtain

$$dL/dc_1 = (L^{(1)} - L)B_i(L^{(1)} - L_0)^{-1}(L^{(1)} - L)(1 - c_1)^{-2}. \quad (5.6)$$

6. Multi-Particle Differential Method (combination with EMM). In the version considered here, it is assumed that the tensor $\tilde{R}_i^*$ (4.2) is determined from the averaged solution to the problem for n inclusions in a uniform medium with tensor properties L and the field $\langle\varepsilon\rangle$ given at infinity. Since the addition of component i (i = 1, 2, ...) at each iteration step of the differential scheme (4.2) is small ($\lambda_i dc/(1 - c) \ll 1$), then it can be assumed that $\langle\hat{\varepsilon}(x)_i\rangle_i$ (3.2), and thus $\tilde{R}_i^*$ as well, can be represented in the form of a power series in the small parameter dc/(1 − c):

$$\tilde{R}_i^* = \sum_{k=0}^{n-1} \lambda^{(k)} \tilde{R}_i^{(k)} \left(\frac{dc}{1-c}\right)^k,\ \lambda^{(k)} = \prod_{i=1}^{n-1} \lambda_i^{k_i},\ \sum_{i=1}^{n-1} k_i = k, \quad (6.1)$$

with $\tilde{R}(^0)_i = \overline{R}_i$. But for each sufficiently smooth function (including for L and $\tilde{R}_i^*$) there is a Taylor expansion in powers of dc/(1 − c). Then, from (4.2) and a comparison of the coefficients of like powers of dc/(1 − c) in the corresponding Taylor series and (6.1), we obtain a ordinary differential equation of order n

$$d^n L/dc^n = \sum_{i=1}^{N} \lambda_i \tilde{R}_i^{(n)} n!/(1-c)^{n+1} \quad (6.2)$$

with initial conditions

$$L^{(0)}(0) = L_0, L^{(1)}(0) = \overline{R}_i, \ldots, L^{(n-1)}(0) = \sum_{i=1}^{N} \tilde{R}_i^{(n-1)}(n-1)! \quad (6.3)$$

7. Multi-Particle Differential Method (combination with EFM). In the proposed method (6.2), $\tilde{R}_i^*$ (4.2) is estimated at each iteration step based on n-particle EFM [1, 3]. It is just this solution for one (2.3), (2.4) and a finite number (2.7) of inclusions located in effective fields $\overline{\varepsilon}(x)$ and $\hat{\varepsilon}(x)_{1,\ldots,n}$, and also the adopted hypothesis H2 [1, 3] $(\langle\widehat{\varepsilon}(x)_{1,\ldots,j,\ldots,n+1}\rangle_i = \langle\widehat{\varepsilon}(x)_{1,\ldots,n}\rangle_i\ (j \neq i,\ 1 \leqslant j \leqslant n,\ x \in v_i))$, which make it possible to close (1.10):

$$\langle\widehat{\varepsilon}(x)_{1,\ldots,j}\rangle_i = \langle\varepsilon\rangle + \int \Big\{\overline{S}(x_i - x_q)\,\varphi(v_q \mid x_1, \ldots, x_j) \sum_{l=1}^{j+1} \overline{Z}_{ql} \overline{R}_l \times \quad (7.1)$$

$$\times\langle\widehat{\varepsilon}(x)_{1,\ldots,j+1}\rangle_l-\overline{S}_i(x_i-x_q)\langle\overline{R}V\widehat{\varepsilon}_1\rangle\Big\}\,dx_q,$$

$$\langle\widehat{\varepsilon}(x)_{1,\ldots,n}\rangle_i=\langle\varepsilon\rangle+\int\Big\{\overline{S}(x_i-x_q)\,\varphi(v_q\,|\,x_1,\ldots,x_j)\sum_{l=1}^{n}\overline{Z}_{ql}\overline{R}_l\langle\widehat{\varepsilon}(x)_{1,\ldots,n}\rangle_l-$$
$$-\overline{S}_i(x_i-x_q)\langle\overline{R}V\widehat{\varepsilon}_1\rangle\Big\}\,dx_q$$

($j = 1, \ldots, n$, $i = 1, \ldots, j$). On the right-hand side of the last equation in (7.1), the tensor $\langle\hat{\varepsilon}_{1,\ldots,n}\rangle$ is formed from the tensor $\langle\hat{\varepsilon}_{1,\ldots,n}\rangle$ on the left-hand side by replacing one of the indices by q. System (7.1) is a linear system of integral equations with respect to $\langle\hat{\varepsilon}(x)_{1,\ldots,j}\rangle_\ell$ ($j = 1, \ldots, n$, $\ell = 1, \ldots, j$). In this system, each line j with $\langle\hat{\varepsilon}(x)_{1,\ldots,j}\rangle_i$ on the left-hand side is made up of j equations, since $i = 1, \ldots, j$. We estimate $\langle\hat{\varepsilon}(x)_{1,\ldots n}\rangle_i$ ($i = 1, \ldots, n$) from the final line in (7.1) by the method of successive approximations for all possible positions of the inclusions $v_1, \ldots, v_n$, and then substitute this into the right-hand side of line $n - 1$, and so on. The estimate $\langle\hat{\varepsilon}(x)_1\rangle_1$ (7.1) makes it possible to determine $\overline{R}^*_i$ (1.5) and $A_0$, $A_i^0$ (1.7). Unlike (6.1), in these relations it has not been assumed that the inclusion concentration is infinitesimal.

We can find an explicit analytical solution to this problem, if we limit the problem to a two-particle approximation and assume that

$$\langle\widehat{\varepsilon}(x)_{12}\rangle_i=\langle\overline{\varepsilon}(x_i)\rangle=\text{const}\ (i=1,2).$$

Then from the first equation in (7.1) we have

$$\overline{R}_i\langle\overline{\varepsilon}_i\rangle=\overline{R}_i\langle\varepsilon\rangle+\overline{R}_i\int\Big\{\overline{S}(x_i-x_q)\sum_{l\neq i,q}^{2}\overline{Z}_{ql}\overline{R}_l\langle\overline{\varepsilon}_l\rangle\,\varphi(v_q\,|\,x_q;x_i)-\tag{7.2}$$
$$-\overline{S}_i(x_i-x_q)\langle\overline{R}\,\overline{\varepsilon}\rangle c_q\Big\}\,dx_q.$$

System (7.2) is a linear algebraic system and is solved for an arbitrary number of components with the assumption that the inclusions relate to the different components, if they have different physical properties, dimensions and orientations. The number of unknowns can be significantly reduced if it is assumed that $\varepsilon_i$ is independent of $\omega_i$. Then averaging (7.2), with certain obvious assumptions, we obtain

$$\langle\!\langle\overline{R}_i\rangle\!\rangle_\omega\langle\overline{\varepsilon}_i\rangle=\sum_{j=1}^{N}(\overline{Y}^{-1})_{ij}\langle\!\langle\overline{R}_j\rangle\!\rangle_\omega\langle\varepsilon\rangle,\tag{7.3}$$

where the matrix $\overline{Y}^{-1}$ has an inverse with elements in the form of a submatrix

$$\overline{Y}_{ij}=\delta_{ij}\Big\{I-\langle\!\langle\overline{R}_i\rangle\!\rangle_\omega\int\langle\!\langle\overline{S}(x_i-x_j)\rangle\!\rangle_\omega\overline{Z}_{ji}\varphi(v_j\,|\,x_q;x_i)\Big\}+\tag{7.4}$$
$$+(\delta_{ij}-1)\langle\!\langle\overline{R}_i\rangle\!\rangle_\omega\int\{\langle\!\langle\overline{S}(x_i-x_j)\rangle\!\rangle_\omega\overline{Z}_{ji}\varphi(v_j\,|\,x_q;v_i)-$$
$$-\langle\!\langle\overline{S}_i(x_i-x_j)\rangle\!\rangle_\omega c_j\Big\}V(x_j;x_i)\,dx_j-\langle\!\langle\overline{R}_i\rangle\!\rangle_\omega\langle\!\langle\overline{P}(v'_{ij})\rangle\!\rangle_\omega c_j.$$

From (1.7) and (7.3) we determine the tensors $\overline{A}_0$, $\overline{A}^0_i$:

$$\overline{A}_0=c_0^{-1}\Big\{I-\sum_{i=1}^{N}c_iA_i\langle\!\langle\overline{R}_i\rangle\!\rangle_\omega^{-1}\sum_{j=1}^{N}(\overline{Y}^{-1})_{ij}\langle\!\langle\overline{R}_j\rangle\!\rangle_\omega\Big\},\tag{7.5}$$
$$\overline{A}_i^0=\overline{B}_i\langle\!\langle\overline{R}_i\rangle\!\rangle_\omega^{-1}\sum_{j=1}^{N}(\overline{Y}^{-1})_{ij}\langle\!\langle\overline{R}_j\rangle\!\rangle_\omega\overline{A}_0^{-1}.$$

In the case of the "quasicrystalline" approximation [2]

$$Z_{ij}=I\delta_{ij},\tag{7.6}$$

where the single-particle method of effective fields is equivalent to the Mori–Tanaka–Eshelby method [2] and (5.3) holds for $A^0_i$, expression (7.4) simplifies:

$$\overline{Y}_{ij} = I\delta_{ij} - \langle\!\langle \overline{R}_i \rangle\!\rangle_\omega \langle\!\langle \overline{P}(v'_{ij}) \rangle\!\rangle_\omega c_j - \langle\!\langle \overline{R}_i \rangle\!\rangle_\omega \int \{\langle\!\langle \overline{S}(x_i - x_j) \rangle\!\rangle_\omega \varphi(v_j \mid x_j; x_i) - \langle\!\langle \overline{S}_i \rangle\!\rangle_\omega c_j\} V(x_j; x_i)\, dx_j. \quad (7.7)$$

Substituting (7.3) into (1.5), we arrive at an expression for the effective modulus as defined by the multi-particle effective field method

$$L^* = L_0 + \sum_{i,j=1}^{N} c_i (Y^{-1})_{ij} \langle\!\langle R_j \rangle\!\rangle_\omega, \quad (7.8)$$

where the tensors Y, $R_j$ are computed, taking into account the equality of the tensor of the properties of the matrix and the tensor $L_0$.

We return to the multi-particle differential EFM in the framework of the scheme for the single-particle method (5.5). According to (7.5), generally speaking, it is not correct to dismiss as unnecessary the Mori–Tanaka hypothesis (5.3), for multi-particle calculation of interacting inclusions. To close (5.2), we must adopt hypothesis (5.4) (or its equivalent), and then switch to an equation analogous to (5.5). It is not possible to draw a strictly similar conclusion for the multi-particle differential EFM, since in using the subsequent expression $\bar{A}^0_i$ (7.5) (and in particular, in (2.5)), a finite inclusion concentration was assumed, although the differential scheme (4.2) is based on infinitesimal additions of the inclusion phase at each iteration step. It is as a result of this contradiction in that there appears in (5.6) an "excess" factor $(L^{(1)} - L_0)^{-1}(L^{(1)} - L)(1 - c)^{-1}$.

This difficulty can be circumvented by assuming that system (7.1), which describes the interaction of n inclusions in the composite medium, is valid for an infinitesimal concentration of the component i ($i = 1, \ldots, N$), equal to $\lambda_i dc/(1 - c)$. This makes it possible to expand $\langle \hat{\varepsilon}(x)_1 \rangle_1$ (7.1) and consequently, $\overline{R}^*_i$ (1.5) as well, in an infinite power series in the small parameter $dc/(1 - c)$:

$$\overline{R}_i^* = \sum_{k=0}^{\infty} \overline{R}_i^{(k)} [dc/(1-c)]^k. \quad (7.9)$$

Unlike (6.1), series (7.9) is infinite even for the n-particle version of EFM (7.1). Comparing (7.9) with the formal Taylor expansion of the tensor $\tilde{R}^*_i$ in terms of $dc/(1 - c)$, we find from (4.2)

$$d^m L/dc^m = \sum_{i=1}^{n} R_i^{(m)} m!/(1-c)^{m+1} \quad (7.10)$$

with Cauchy conditions

$$L^{(0)}(0) = L_0, L^{(1)}(0) = R_i, \ldots, L^{(m-1)}(0) = R_i^{(m-1)}(m-1)!$$

In general, the order of (7.10) is larger than the number of interacting inclusions ($m \geq n$). For $m = n > 2$, the estimates made using the differential methods EMM (6.2) and EFM (7.10) are different.

8. Example. We examine a linear-elastic problem for a composite medium with an incompressible isotropic matrix, filled with rigid spherical inclusions of one size (N = 1). Then, from (2.4) and (7.4), we have [1]

$$\overline{R}_1 = \frac{5}{2}\mu,\ Y_{11} = 1 - 31c/16 \quad (8.1)$$

($\mu$ is the shear component of the isotropic tensor L). In deriving the second equation in (8.1) we applied the approximation for matrix $Z_{ij}$ (2.9) with $S(x_i - x_j) = U(x_i - x_j)$, $\phi(v_j|x_j; x_i) = \phi(v_j) = n_j$ for $x_j \notin v_i^0$. We obtain the following relations for the effective shear modulus of the composite $\mu^*$ as a function of the shear modulus of the matrix $\mu_0$:

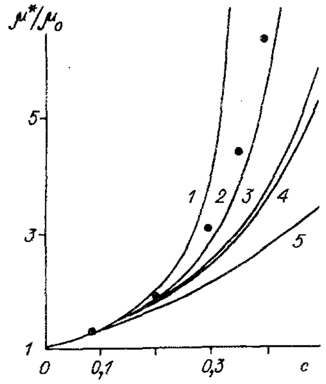

Fig. 1

$$\mu^* = \mu_0(1 - 5c/2)^{-1}; \tag{8.2}$$

$$\mu^* = \mu_0(1 + 3c/2)(1 - c)^{-1}; \tag{8.3}$$

$$\mu^* = \mu_0\{1 - 5c(1 + 31c/16)/2\}^{-1}; \tag{8.4}$$

$$\mu^* = \mu_0(1 - c)^{-5/2}. \tag{8.5}$$

Formula (8.2) is determined using single-particle EMM (3.1); (8.3) is determined using single-particle EFM (7.7), (7.8). Formula (8.4) is from two-particle EMM (3.2); and (8.5) is from single-particle differential method EMM (4.2), (5.1). The differential equation

$$\frac{d^2\mu}{dc^2} = \frac{155}{16}\mu(1 - c)^{-2}, \tag{8.6}$$

$$\mu(0) = \mu_0,\ \mu^{(1)}(0) = \frac{5}{2}\mu_0$$

corresponds to two-particle differential EMM (6.2) and EFM (7.10) with m = 2 (7.10), while the equation

$$d^3\mu/dc^3 = 15\left(\frac{31}{16}\right)^2 \mu(1 - c)^{-3},\ \mu(0) = \mu_0,\ \mu^{(1)}(0) = \frac{5}{2}\mu_0,\ \mu^{(2)}(0) = \frac{155}{16}\mu_0$$

corresponds to two-particle differential EFM (7.10) with m = 3. Figure 1 shows the experimental data (points) [19], and curves 1-5, computed from Eqs. (8.2), (7.10) with m = 30, (8.6), (8.5), and (8.3), respectively. It is clear that taking the binary interaction of the inclusions into account can raise the accuracy of the differential methods.

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# EQUILIBRIUM AND STABILITY OF A NONLINEAR-ELASTIC PLATE WITH A TAPERED DISCLINATION

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UDC 539.3

In solid-state physics, it is important to study dislocations and disclinations in two-dimensional bodies (plates, films, and so forth), as well as in three-dimensional bodies. Methods of analyzing such problems and specific solutions are given, for example, in [1-3]. One of these defects arises in the study, introduced in [4], of a tapered disclination, a stress–strain state in a cylinder which is made up of a large number of thin disks which contain disclinations. In this case, the disks do not remain plane, but are transformed into either conical funnels or complex curved surfaces with a saddle-shaped configurations. These were observed in numerical simulation of disclinations using molecular dynamics methods [4]. This has generated interest in the study of similar models using the methods of elasticity theory. The linear theory of dislocations in shells is explained in [5], and its nonlinear aspects, in [6].

In this work, the problem of equilibrium and stability of a nonlinear-elastic plate with tapered disclinations is studied. The analysis is based on nonlinear equations of the theory of Love plates and shells, formulated in [7]. In the framework of momentless theory, it is established that in the case of positive disclination, two axisymmetric equilibrium shapes are possible: equilibrium plane or conic surfaces. A nonaxisymmetric nonplane equilibrium form for the momentless plate with negative discilination is determined. The equilibrium equations of nonlinear momentum theory admit a "plane" solution (deflection identically equal to zero) for all values of the disclination parameter, which coincides with the solution from momentless theory. A numerical investigation of its stability has been done.

1. We consider a plate of thickness h, having the form of a ring $c \leq r \leq d$, $0 \leq \phi \leq 2\pi$, $-h/2 \leq \zeta \leq h/2$ ($r$, $\phi$, $\zeta$ are cylindrical coordinates). We adopt the Kirchoff–Love hypothesis, and further, by plate, we will understand its average surface. The formation of a disclination in the plates is given by the relations

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